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Applications of coverings and non-local symmetries

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Abstract. Bluman, Reid and Kumei have introduced the potential symmetries, i.e. special non-local symmetries. The concept of a non-local symmetry is aesthetically justified by the theory of coverings. The basic idea is that solutions of the covering system imply solutions of the covered system. Bluman, Kumei and Reid never considered the symmetries lost, i.e. local symmetries of the covered system that do not extend to local symmetries of the covering system. Here, examples of coverings are given that recover these lost symmetries, demonstrating the more natural setting of coverings.

1. Introduction

Bluman and Reid (1988) and Bluman and Kumei (1989) have enlarged the set of local symmetries by introducing the potential symmetries. A potential is an auxiliary dependent variable that corresponds to a conserved form. A system of which one of the equations is written in conserved form, gives rise to an auxiliary system which includes the potential. A potential symmetry of a system is defined as a local symmetry of the auxiliary system that does not correspond to a local symmetry of the system itself.

On the one hand potential symmetries truly enlarge the set of symmetries of a system. On the other they involve a strange phenomenon, here called the loss of symmetries. Loss of symmetries can occur, as local symmetries of the system do not have to extend to local symmetries of its auxiliary system.

This loss is disposed of by the introduction of non-local symmetries. The notion of these symmetries is based on the theory of coverings developed by Krasil'shchik and Vinogradov (1980, 1984a, b, 1989). Potential symmetries are special cases of non-local symmetries. In this paper examples of coverings that recover lost symmetries presented in Bluman and Reid (1988) and Bluman and Kumei (1989) are given.

In section 2 local symmetries are introduced. In section 3 the theory of coverings is explained briefly and the non-local symmetries are introduced. In section 4 an introduction to the examples of coverings is given. Four examples of coverings that recover lost symmetries have been included in sections 5–8. In section 9 we discuss the geometric framework of coverings.

2. Symmetries

Several authors (e.g. Ibragimov (1989), Olver (1987), Ovsiannikov (1982), and Vinogradov (1984)) discussed the various aspects of symmetries of differential equations. Here, the notion of a symmetry is explained following Olver's work.

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In differential geometric theory a system of partial differential equations is defined as an algebraic submanifold of some finite-order jet bundle. We will explain this for an nth-order system Δ involving p independent variables $x = (x_1, \ldots, x_p)$ and q dependent variables $u = (u^1, \ldots, u^q)$. Multi-indices $I = (i_1, \ldots, i_p)$ enable a compact notation for derivatives of u: $u_I^\alpha \equiv \partial^{|I|}u^\alpha/\partial x^I = \partial^{i_1+\cdots+i_p}u^\alpha/\partial x_1^{i_1}\cdots\partial x_p^{i_p}$, e.g. $u^2_{(2,0,1)} = \partial^3 u^2/\partial x_1^2 \partial x_3$. The kth-order *jet bundle* J^k *is a manifold with local coordinate representation* $(x, u_I)_{|I|\leqslant k}$. The nth-order system Δ is the submanifold $\{F = 0\} \subset J^n$, e.g. the Korteweg–DeVries equation is given by $\{u_t = uu_x + u_{xxx}\} \subset J^3 = \{(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, u_{xxx}, u_{xxt}, u_{xtt}, u_{ttt})\}$. We will use the infinite jet space J^∞ too.

The algebraic analogue of the partial derivative $\partial/\partial x_k$ is given by the total derivative D_k (Olver 1987, p 112)

$$\mathrm{D}_k = \frac{\partial}{\partial x_k} + \sum_{\alpha=1}^{q}\sum_{|I|\geqslant 0} u^\alpha_{I,k}\frac{\partial}{\partial u^\alpha_I} \quad k = 1, \ldots, p \tag{2.1}$$

where $u^\alpha_{(I,k)}$ is the variable corresponding to $\partial^{|I|+1}u^\alpha/\partial x^I \partial x_k$. Likewise, the higher-order partial derivative $\partial^{|I|}/\partial x^I$ is represented in the algebraic setting by $\mathrm{D}^I = \mathrm{D}_1^{i_1} \circ \cdots \circ \mathrm{D}_p^{i_p}$ where $\mathrm{D}_k^i = \mathrm{D}_k^{i-1} \circ \mathrm{D}_k$. Differential consequences of the nth-order system $\Delta = \{F = 0\} \subset J^n$ are given by prolongation. The kth-order prolongation of the system Δ is given by $\mathrm{pr}^{(k)}\Delta = \{\mathrm{pr}^{(k)}F = 0\} \subset J^{k+n}$ with $\mathrm{pr}^{(k)}F = \left(\mathrm{D}^I F\right)_{|I|\leqslant k}$. The infinite prolongation of Δ is denoted by $\mathrm{pr}\Delta$ ($= \mathrm{pr}^\infty\Delta \subset J^\infty$).

Roughly speaking, symmetries of a system are one-parameter groups of transformations that transform solutions of the system to other solutions of the same system. A one-parameter group of transformation is directly related to its infinitesimal generator, i.e. a vector field. From now on, a symmetry is identified with its infinitesimal generator. The infinitesimal generators are fixed by fixing their components on J^0 (Olver 1987, p 295):

$$v = \sum_{i=1}^{p}\xi^i\frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q}\varphi_\alpha\frac{\partial}{\partial u^\alpha}. \tag{2.2}$$

The coefficients ξ^i and φ_α are functions on some finite-order jet space. Other components of the infinitesimal generator are given by prolongation of v; the prolongation formula reads (Olver 1987, p 295)

$$\mathrm{pr}^{(k)}v = v + \sum_{\alpha=1}^{q}\sum_{|I|=1}^{k} Q^I_\alpha\frac{\partial}{\partial u^\alpha_I} \qquad Q^I_\alpha = \mathrm{D}^I\left(\varphi_\alpha - \sum_{i=1}^{p}\xi^i u^\alpha_i\right) + \sum_{i=1}^{p}\xi^i u^\alpha_{I,i}. \tag{2.3}$$

Because ξ^i and φ_α are defined on some finite order jet bundle, so are the Q^I_α. Now we give the symmetry equation.

Definition 2.1 (Olver 1987, p 296). $v = \sum_{i=1}^p \xi^i\partial/\partial x_i + \sum_{\alpha=1}^q \varphi_\alpha\partial/\partial u^\alpha$, with $\xi^i, \varphi_\alpha \colon J^k \to \mathbb{R}$, corresponds to a symmetry of $\Delta = \{F = 0\} \subset J^n$ if and only if

$$\mathrm{pr}^{(n)}v(F) = 0 \quad \text{whenever} \quad \mathrm{pr}^{(k)}F = 0. \tag{2.4}$$

The symmetries defined here are known as local or generalized symmetries. Generalized, because they generalize the point symmetries, which are symmetries v that are well defined vector fields on J^0, i.e. ξ^i and φ_α are functions on J^0.

Trivial symmetries (Olver 1987, p 298) give rise to equivalence classes of symmetries and suitable representatives are given by the evolutionary vector fields (Olver 1987, p 297)

$$v_Q = \sum_{\alpha=1}^{q} Q_\alpha \frac{\partial}{\partial u^\alpha} \tag{2.5}$$

of which the characteristic $Q = (Q_1, \ldots, Q_q)$ is defined on the algebraic set $\mathrm{pr}^{(k)}\Delta \subset J^{k+n}$ for some k. The representative of the equivalence class corresponding to $v = \sum_{i=1}^{p} \xi^i \partial/\partial x_i + \sum_{\alpha=1}^{q} \varphi_\alpha \partial/\partial u^\alpha$ has characteristic $Q_\alpha = \varphi_\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha$, $\alpha = 1, \ldots, q$.

3. Coverings and non-local symmetries

The theory of coverings forms the differential geometric setting that allows us to introduce non-local symmetries. These symmetries arise, if the characteristic Q of the evolutionary vector field v_Q permit dependency on non-local variables, i.e. integrals. Interesting cases concern non-local variables that correspond to integrability conditions of the system.

The idea behind the theory of coverings is that systems of partial differential equations can cover each other, i.e. solutions of the covering system give rise to solutions of the covered system. The covering system can be viewed as the covered system extended with non-local variables. Krasil'shchik and Vinogradov have succeeded in giving concrete form to this idea and in this section their work (Krasil'shchik and Vinogradov 1989) is followed. Earlier publications on a exposition of their idea (Krasil'shchik and Vinogradov 1984a, b, 1989) might be easier to understand. Before the theory of coverings is discussed, the algebraic concept of a system of partial differential equations, as has been introduced in the previous section, needs reconsideration.

The infinite prolonged system $\mathrm{pr}\Delta = \{\mathrm{pr}F = 0\} \subset J^\infty$ is assumed to be a regular manifold itself. Local coordinates of $\mathrm{pr}\Delta$ are selected from the local coordinates of J^∞; the selected ones are called internal coordinates. To enhance local coordinate representation, the following sets of multi-indices are introduced: $\mathcal{I}(\alpha) = \{I : u_I^\alpha$ is internal coordinate$\}$ for $\alpha = 1, \ldots, q$. Thus the local coordinate representation of $\mathrm{pr}\Delta$ is given by $\{(x, u_I^\alpha)_{I \in \mathcal{I}(\alpha)}\}$. The other local coordinates of J^∞ are assumed to be expressed in terms of the internal coordinates.

The total derivatives D_k corresponding to J^∞, $k = 1, \ldots, p$, are restricted to $\mathrm{pr}\Delta$; the restriction of D_k is denoted by $\bar{\mathrm{D}}_k$:

$$\bar{\mathrm{D}}_k = \frac{\partial}{\partial x_k} + \sum_{\alpha=1}^{q} \sum_{I \in \mathcal{I}(\alpha)} u_{I,k}^\alpha \frac{\partial}{\partial u_I^\alpha} \qquad k = 1, \ldots, p. \tag{3.1}$$

Note that if $u_{I,k}^\alpha$ is not an internal coordinate, i.e. $(I, k) \notin \mathcal{I}(\alpha)$, then this coordinate is expressed in terms of the internal coordinates.

The differential equation Δ is now seen as the manifold $\mathrm{pr}\Delta$ provided with Cartan distribution generated by $\bar{\mathrm{D}}_1, \ldots, \bar{\mathrm{D}}_p$, i.e. each point of $\mathrm{pr}\Delta$ is linked up with the linear space generated by $\bar{\mathrm{D}}_1, \ldots, \bar{\mathrm{D}}_p$. Essential features of the Cartan distribution are its finite dimension, i.e. p, and its complete integrability, i.e. $\bar{\mathrm{D}}_i$ and $\bar{\mathrm{D}}_j$ commute ($[\bar{\mathrm{D}}_i, \bar{\mathrm{D}}_j] = 0$) for $i, j = 1, \ldots, p$.

The theory of coverings is based on the concept of a system of partial differential equations as a manifold provided with a Cartan distribution. A simplified, but strong enough definition of a covering is the following.

Definition 3.1 (Krasil'shchik and Vinogradov 1989, p 166). A covering of the system prΔ is given by a projection $\tau: A \to \mathrm{pr}\Delta$, whereby A is a manifold provided with a Cartan distribution, such that the Cartan distribution of A is projected onto the Cartan distribution of prΔ. In other words A covers prΔ.

Coverings of a system prΔ are found by searching for extensions of the form $\widetilde{\mathrm{pr}}\Delta = \mathrm{pr}\Delta \times \mathbb{R}^s$, where s may be infinite. In section 9 we discuss the reason for taking infinite-dimensional coverings into consideration as well. In local coordinates $\widetilde{\mathrm{pr}}\Delta$ looks like $\{(x, u_I^\alpha)_{I\in\mathcal{I}(\alpha)} \times (\omega^1, \ldots, \omega^s)\}$ where $\omega = (\omega^1, \ldots, \omega^s)$ are the non-local variables. The total derivatives have to be extended by the components for ω:

$$\widetilde{\mathrm{D}}_k = \bar{\mathrm{D}}_k + \sum_{\alpha=1}^{s} X_\alpha^k \frac{\partial}{\partial \omega_\alpha} \qquad k = 1, \ldots, p. \tag{3.2}$$

The coefficients X_α^k, of which each depends on a finite number of variables in $\widetilde{\mathrm{pr}}\Delta$ only, have to satisfy the integrability conditions $[\widetilde{\mathrm{D}}_i, \widetilde{\mathrm{D}}_j] = 0$, also written as

$$\widetilde{\mathrm{D}}_i(X_\alpha^j) = \widetilde{\mathrm{D}}_j(X_\alpha^i) \qquad i, j = 1, \ldots, p \qquad \alpha = 1, \ldots, s. \tag{3.3}$$

These equations are also known by the work of Estabrook and Wahlquist (1975) about prolongation structures of differential equations.

Some integrability conditions provide no information about the system at all and the corresponding covering is therefore called trivial (Krasil'shchik and Vinogradov, p 169). This gives rise to the following equivalence classes: isomorphic coverings are called equivalent. When describing coverings over prΔ, it is natural to consider the equivalence classes only.

Once a covering $\tau: \widetilde{\mathrm{pr}}\Delta \to \mathrm{pr}\Delta$ has been established, one might wonder if $\widetilde{\mathrm{pr}}\Delta$ corresponds to a system of partial differential equations, i.e. can be written as $\mathrm{pr}\widetilde{\Delta}$. The answer is positive; the covering system $\widetilde{\Delta}$ is given by (Krasil'shchik and Vinogradov 1989, p 168)

$$\Delta \qquad w_i^\alpha = X_\alpha^i \qquad i = 1, \ldots, p \qquad \alpha = 1, \ldots, s \tag{3.4}$$

where w_i^α represents the derivative $\partial w^\alpha / \partial x_i$. The system $\widetilde{\Delta}$ is well defined, for in algebraic sense the integrability equations $\partial^2 w^\alpha / \partial x_i \partial x_j = \partial^2 w^\alpha / \partial x_j \partial x_i$ are equivalent to (3.3) and so correspond to differential consequences of the system Δ. Solutions of $\widetilde{\Delta}$ give rise to solutions of Δ and therefore $\widetilde{\Delta}$ is said to cover Δ.

Definition 3.2 (Krasil'shchik and Vinogradov 1989, p 185). Symmetries of the covering system are called non-local symmetries of the covered system.

The projection of the non-local symmetry, e.g. $v_{(Q,R)} = \sum_{\alpha=1}^{q} Q_\alpha \partial/\partial u^\alpha + \sum_{\alpha=1}^{s} R_\alpha \partial/\partial w^\alpha$ is called the shadow (Krasil'shchik and Vinogradov 1989, p 185), i.e. $v_Q = \sum_{\alpha=1}^{q} Q_\alpha \partial/\partial u^\alpha$. The shadow of a non-local symmetry is a non-local solution of the covered system.

Definition 3.3 (Krasil'shchik and Vinogradov 1989, p 195). A non-local solution of the system $\mathrm{pr}\Delta = \{\mathrm{pr}F = 0\}$ covered by $\widetilde{\mathrm{pr}}\Delta = \mathrm{pr}\Delta \times \omega$, is given by a characteristic Q defined on $\widetilde{\mathrm{pr}}\Delta$, which satisfies

$$\widetilde{\mathrm{pr}}\, v_Q(F) = 0 \tag{3.5}$$

where $\widetilde{\mathrm{pr}}\, v_Q = \sum_{\alpha=1}^{q} \sum_{|I|=0}^{\infty} \widetilde{\mathrm{D}}^I(Q_\alpha) \partial/\partial u_I^\alpha$.

For convenience the evolutionary vector field v_Q corresponding to a non-local solution Q is also referred to as a non-local solution. A shadow is a non-local solution, but is a non-local solution a shadow? In other words, does there exist a covering in which the non-local solution is the shadow of a local symmetry? The answer to this question has been given by Khor'kova (1989) in the following theorem.

Theorem 3.1 (Krasil'shchik and Vinogradov 1989, p 198). To any non-local solution Q corresponds a non-local symmetry in an appropriate extension of the covering system.

In section 7 a sketch of the proof to this theorem is revealed; see also section 9. Examples of this theorem are found in sections 5, 6 and 8.

4. Introduction to the examples

Examples of coverings are given in the light of examples taken from the explanation of potential symmetries presented by Bluman and and Kumei (1989) and Bluman and Reid (1988). Potential symmetries of a system, of which one of the equations is in conserved form, are local symmetries of the auxiliary system that arises by introduction of the potential corresponding to the conserved form. The auxiliary system covers the original system and so potential symmetries are special cases of non-local symmetries.

We will show that the disappearance of symmetries, i.e. local symmetries of the original system that do not correspond to local symmetries of the auxiliary system, is disposed of, for any symmetry lost is recovered in an appropriate covering of the auxiliary system. In fact the lost symmetries are very special non-local solutions of the given covering and so theorem 3.1 is applicable.

Bluman and others have computed point symmetries of the wave equation $u_{yy} = c^2 u_{xx}$, in which the wave speed $c = c(x)$, and of the auxiliary system corresponding to the potential v: $v_x = u_y/c^2$ and $v_y = u_x$. In the first example we set c equal to $\sqrt{x}$. In this setting one symmetry is lost, but recovered in an infinite-dimensional covering.

In the other three examples c equals $1 - x^2$, in which case two point symmetries are lost. In the second examnle one of them is recovered in a one-dimensional extension of the auxiliary system.

In the third example the other lost symmetry is recovered. Computations similar to the ones performed in the previous examples are seen to be more complicated. Therefore an abstract construction of the desired covering, which runs in accordance with the sketch of the proof to theorem 3.1, is favoured.

In the fourth example a covering of the auxiliary system is given, not constructed, which recovers the lost symmetry considered in the previous example.

5. First example

Bluman has computed the point symmetries of the wave equation

$$u_{yy} = x u_{xx} \tag{5.1}$$

and the auxiliary system which includes the potential v:

$$u_{yy} = x u_{xx} \qquad v_x = u_y/x \qquad v_y = u_x \,. \tag{5.2}$$

The point symmetry of the covered system (5.1), which cannot be extended to a local symmetry of the covering system (5.2), is given by

$$X = \big(yu - 4xyu_x - (y^2+4x)u_y\big)\partial_u \qquad \big(\equiv 4xy\partial_x + (y^2+4x)\partial_y + yu\partial_u\big) \tag{5.3}$$

of which the u-component is denoted by $U = yu - 4xyu_x - (y^2+4x)u_y$.

The attempt to extend X to a symmetry of system (5.2), involves the introduction of the v-component, whose coefficient, V, has to satisfy

$$\begin{aligned} \mathrm{D}_x(V) &= \mathrm{D}_y(U)/x = \big(yu_y + u - 4xyu_{xy} - 4xu_x - (y^2+4x)u_{yy} - 2yu_y\big)/x \\ \mathrm{D}_y(V) &= \mathrm{D}_x(U) = yu_x - 4xyu_{xx} - 4yu_x - (y^2+4x)u_{xy} - 4u_y\,. \end{aligned} \tag{5.4}$$

The second equation of (5.4) reduces to $\mathrm{D}_y(V) = \mathrm{D}_y(-yv - 4xyv_x - (y^2+4x)v_y) + v$ and can be solved only if the integral $u^1 = \int v\,\mathrm{d}y$ is introduced. In fact u^1 is the potential corresponding to the differential equation $v_x = u_y/x$:

$$u_{yy} = xu_{xx} \qquad v_x = u_y/x \quad v_y = u_x \qquad u^1_x = u/x \quad u^1_y = v\,. \tag{5.5}$$

The variables present in the setting (5.5) enable solving (5.4), whose solution is $V = -yv - 4xyv_x - (y^2+4x)v_y + u^1$. To be precise, the tuple (U, V) is a non-local solution of the system (5.2) covered by (5.5).

Next the u^1-component of X needs to be computed, whose coefficient has to satisfy the following system:

$$\begin{aligned} \mathrm{D}_x(U^1) &= U/x = \big(yu - 4xyu_x - (y^2+4x)u_y\big)/x \\ \mathrm{D}_y(U^1) &= V = -yv - 4xyv_x - (y^2+4x)v_y + u^1\,. \end{aligned} \tag{5.6}$$

The solution of the first equation is to be $U^1 = yu^1 - 4xyu^1_x - (y^2+4x)u^1_y + 4\int v\,\mathrm{d}x$. Also the integral $v^1 = \int v\,\mathrm{d}x$ is a potential, giving rise to

$$u_{yy} = xu_{xx} \qquad v_x = u_y/x \quad v_y = u_x \qquad u^1_x = u/x \quad u^1_y = v \qquad v^1_x = v \quad v^1_y = u\,. \tag{5.7}$$

The non-local solution $(U,\ V,\ U^1)$, where $U^1 = yu^1 - 4xyu^1_x - (y^2+4x)u^1_y + 4v^1$, of system (5.5) covered by system (5.7) is to be extended by the v^1-component, whose coefficient has to satisfy

$$\begin{aligned} \mathrm{D}_x(V^1) &= V = -yv - 4xyv_x - (y^2+4x)v_y + u^1 \\ \mathrm{D}_y(V^1) &= U = -yu - 4xyu_x - (y^2+4x)u_y\,. \end{aligned} \tag{5.8}$$

To solve this system, the potential $u^2 = \int v^1\,\mathrm{d}y$ is added, the superscript of which is an index, not a power. Including the potential u^2 the system (5.7) extends to

$$\begin{aligned} &u_{yy} = xu_{xx} \qquad v_x = u_y/x \quad v_y = u_x \qquad u^1_x = u/x \quad u^1_y = v \\ &v^1_x = v \quad v^1_y = u \qquad u^2_x = u^1 \quad u^2_y = v^1 \end{aligned} \tag{5.9}$$

and the solution of (5.8) reads $V^1 = 3yv^1 - 4xyv^1_x - (y^2+4x)v^1_y + 4xu^2_x - 3u^2$.

Likewise computation of the u^2-component, whose coefficient is to be computed from $D_x(U^2) = U^1$ and $D_y(U^2) = V^1$, involves yet another potential v^2:

$$
\begin{aligned}
&u_{yy} = xu_{xx} \qquad v_x = u_y/x \quad v_y = u_x \qquad u^1_x = u/x \quad u^1_y = v \\
&v^1_x = v \quad v^1_y = u \qquad u^2_x = u^1 \quad u^2_y = v^1 \qquad v^2_x = v^1 \quad v^2_y = xu^1 - u^2 .
\end{aligned}
\tag{5.10}
$$

The solution $U^2 = 5yu^2 - 4xyu^2_x - (y^2+4x)u^2_y + 8v^2$ turns $(U,\ V,\ U^1,\ V^1,\ U^2)$ into a non-local solution of system (5.9) covered by (5.10).

The coefficient of the v^2-component, denoted by V^2, has to satisfy $D_x(V^2) = V^1$ and $D_y(V^2) = xU^1 - U^2$ and is found to be expressed in another potential u^3, $V^2 = 7yv^2 - 4xyv^2_x - (y^2+4x)v^2_y + 8xu^3_x - 15u^3$. Including the potential u^3, the system (5.10) extends to

$$
\begin{aligned}
&u_{yy} = xu_{xx} \qquad v_x = u_y/x \quad v_y = u_x \qquad u^1_x = u/x \quad u^1_y = v \qquad v^1_x = v \quad v^1_y = u \\
&u^2_x = u^1 \quad u^2_y = v^1 \qquad v^2_x = v^1 \quad v^2_y = xu^1 - u^2 \qquad u^3_x = u^2 \quad u^3_y = v^2
\end{aligned}
\tag{5.11}
$$

and so on. An infinite number of variables $u, v, u^1, v^1, u^2, v^2, \ldots$ evolve, which are seen to satisfy the system

$$
\begin{aligned}
&u_{yy} = xu_{xx} \qquad v_x = u_y/x \quad v_y = u_x \qquad u^1_x = u/x \quad u^1_y = v \\
&v^i_x = u^i_y \quad v^i_y = xu^i_x - (i-1)u^i \qquad u^{i+1}_x = u^i \quad u^{i+1}_y = v^i \qquad i = 1, 2, 3, \ldots .
\end{aligned}
\tag{5.12}
$$

In the setting of this system the symmetry X (5.3) prolongs to the symmetry $U\partial_u + V\partial_v + \sum_{i=1}^{\infty}(U^i\partial_{u^i} + V^i\partial_{v^i})$, where

$$
\begin{aligned}
U &= yu - 4xyu_x - (y^2+4x)u_y \\
V &= -yv - 4xyv_x - (y^2+4x)v_y + u^1 \\
U^i &= (4i-3)yu^i - 4xyu^i_x - (y^2+4x)u^i_y + 4iv^i \\
V^i &= (4i-1)yv^i - 4xyv^i_x - (y^2+4x)v^i_y + 4ixu^{i+1}_x - (4i^2-1)u^{i+1} .
\end{aligned}
\tag{5.13}
$$

Thus the lost symmetry of system (5.1) covered by system (5.2) turns out to be a non-local symmetry of (5.2).

6. Second example

Here one of the two lost point symmetries of the wave equation

$$
u_{yy} = c^2 u_{xx} \qquad c = 1 - x^2 \tag{6.1}
$$

covered by the auxiliary system

$$
u_{yy} = c^2 u_{xx} \qquad v_x = u_y/c^2 \quad v_y = u_x \tag{6.2}
$$

is considered, whereas the other is treated in the next subsection.

The symmetry $Y = U\partial_u$ of the wave equation (6.1)

$$Y = (xu + cu_x)\partial_u \tag{6.3}$$

needs to be extended by $V\partial_v$, such that $U\partial_u + V\partial_v$ is a symmetry of the auxiliary system (6.2). The symmetry equation results in the following system for V:

$$\begin{aligned} D_x(V) &= D_y(U)/c^2 = (xu_y + cu_{xy})/c^2 = D_x(-xv + cv_x) + v \\ D_y(V) &= D_x(U) = (u - xu_x + cu_{xx}) = D_y(-xv + cv_x) + u\,. \end{aligned} \tag{6.4}$$

In order to solve this system for V, the integral $\int v\,dx$ or $\int u\,dy$ needs to be available. Both integrals equal the potential w corresponding to the differential equation $v_y = u_x$, giving rise to the extended system

$$u_{yy} = c^2 u_{xx} \qquad v_x = u_y/c^2 \quad v_y = u_x \qquad w_x = v \quad w_y = u\,. \tag{6.5}$$

and the solution of (6.4) reads $V = -xv + cv_x + w$.

Next the non-local solution $(U,\ V)$ of system (6.2) covered by (6.5) is to be extended to a symmetry $U\partial_u + V\partial_v + W\partial_w$ of the covering system, where W is to be computed from

$$\begin{aligned} D_x(W) &= V = -xv + cv_x + w = D_x(xw + cw_x) \\ D_y(W) &= U = xu + cu_x = D_y(xw + cw_x)\,. \end{aligned} \tag{6.6}$$

As it happens, this system for W can be solved without introduction of further non-local variables, the solution being $W = xw + cw_x$. So the symmetry Y (equation (6.3)) prolongs to the symmetry of system (6.5) given by

$$(xu + cu_x)\partial_u + (-xv + cv_x + w)\partial_v + (xw + cw_x)\partial_w \tag{6.7}$$

which in consequence is a non-local symmetry of system (6.2).

7. Third example

In the previous section one of the two lost point symmetries has been recovered. Here the other is considered.

The symmetry $Z = U\partial_u$ of system (6.1)

$$Z = (xyu + cyu_x + Cu_y)\partial_u \qquad C = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right) \tag{7.1}$$

is to be extended to a symmetry $U\partial_u + V\partial_v$ of system (6.2), so V has to satisfy

$$\begin{aligned} D_x(V) &= D_y(U)/c^2 = (xu + xyu_y + cu_x + cyu_{xy} + Cu_{yy})/c^2 \\ D_y(V) &= D_x(U) = yu + xyu_x - 2xyu_x + cyu_{xx} + u_y/c + Cu_{xy}\,. \end{aligned} \tag{7.2}$$

The solution to these equation reads $V = -xyv + cyv_x + Cv_y + w$, where w is a potential of system (6.2), giving rise to the extended system

$$u_{yy} = c^2 u_{xx} \qquad v_x = u_y/c^2 \quad v_y = u_x \qquad w_x = xu/c^2 + yv \quad w_y = yu + xv\,. \tag{7.3}$$

Next the w-component is to be computed from

$$\begin{aligned} \mathrm{D}_x(W) &= xU/c^2 + yV = \mathrm{D}_x(xyw + cyw_x + Cw_y) - \big(2yu/c^2 + (C + x/c)v\big) \\ \mathrm{D}_y(W) &= yU + xV = \mathrm{D}_y(xyw + cyw_x + Cw_y) - \big((C + x/c)u + 2yv\big) \end{aligned} \tag{7.4}$$

whose solution is $W = xyw + cyw_x + Cw_y - p$, including the non-local variable p, leading to the extension

$$(7.3) \qquad p_x = 2yu/c^2 + (C + x/c)v \qquad p_y = (C + x/c)u + 2yv\,. \tag{7.5}$$

Likewise the non-local solution $(U,\ V,\ W)$ of system (7.3) covered by (7.5) has to be turned into a symmetry $U\partial_u + V\partial_v + W\partial_w + P\partial_p$ of (7.5), so P is to be computed from

$$\mathrm{D}_x(P) = 2yU/c^2 + (C + x/c)V \qquad \mathrm{D}_y(P) = (C + x/c)U + 2yV\,. \tag{7.6}$$

The solution $P = (C + x/c)(xw + cyv + Cu) + 2Cyv + 2y^2u/c - 4w/c + q$ has given rise to the extension including q:

$$(7.5) \qquad q_x = (3x/c - 3C - 2xy^2)u/c^2 + 6xw/c^2 \qquad q_y = (3x/c - 3C - 2xy^2)v + 2yw\,. \tag{7.7}$$

Further extensions of system (7.7) are necessary for construction of the covering of the auxiliary system (6.2) in which the symmetry Z (7.1) is extended properly, but the computations involved are hard to perform and therefore the abstract approach as given in the sketch of the proof to theorem 3.1 is preferred. For the case under consideration this boils down to a series of successively one-dimensional extensions of the covering system, in which limit the non-local solution prolongs to a symmetry. The construction uses induction on the number of potentials introduced.

Suppose that after several one-dimensional extensions, whereby the potentials $p^1, \ldots,$ p^{n-1} have been included, the non-local solution $v_{n-1} = v_{Q_{n-1}}$ of the system Δ_{n-1} covered by the system Δ_n has been constructed. Consider the covering system Δ_n arisen from Δ_{n-1} by including the potential p^n:

$$\Delta_n = \begin{cases} \Delta_{n-1} \\ p^n_x = F^{n-1},\ p^n_y = G^{n-1}\,. \end{cases} \tag{7.8}$$

Thus the integrability equation $p^n_{xy} = p^n_{yx}$ gives rise to the conservation law $\mathrm{D}_y(F^{n-1}) = \mathrm{D}_x(G^{n-1})$ of Δ_{n-1}.

The attempt to extend the non-local solution $v_{n-1} = U\partial_u + \sum_{i=1}^{n-1} P^i\partial_{p^i}$ to a symmetry $v_{n-1} + P^n\partial_{p^n}$ of Δ_n results in the following system for P^n:

$$\mathrm{D}_x(P^n) = F^n \overset{\text{def}}{=} \operatorname{pr} v_{n-1}(F^{n-1}) \qquad \mathrm{D}_y(P^n) = G^n \overset{\text{def}}{=} \operatorname{pr} v_{n-1}(G^{n-1}) \tag{7.9}$$

giving rise to the definitions of F^n and G^n.

Assume that this system for P^n cannot be solved in the current setting, otherwise the desired covering has been provided, i.e. the non-local solution $v_0 = U\partial_u$ of Δ_0 covered by Δ_1 extends to the symmetry $v_n = v_{n-1} + P^n\partial_{p^n}$ of Δ_n.

The solution of system (7.9) may then be given by $P^n = p^{n+1}$, where the potential p^{n+1} corresponds to the conservation law $D_y(F^n) = D_x(G^n)$ on Δ_n (Krasil'shchik and Vinogradov 1984, p 204). So v_{n-1} extends to the non-local solution $v_n = v_{n-1} + p^{n+1}\partial_{p^n}$ of Δ_n covered by Δ_{n+1}:

$$\Delta_{n+1} = \begin{cases} \Delta_n \\ p_x^{n+1} = F^n,\ p_y^{n+1} = G^n . \end{cases} \tag{7.10}$$

Next, the same has to be done for P^{n+1} and so on. By induction, an infinite number of potentials is introduced, giving rise to the system

$$\widetilde{\Delta} = \begin{cases} \Delta_0 \\ p_x^{i+1} = F^i,\ p_y^{i+1} = G^i,\ i = 0, 1, 2, \ldots . \end{cases} \tag{7.11}$$

The system $\widetilde{\Delta}$ covers Δ_1 as desired, for the non-local solution $v_0 = U\partial_u$ corresponds to the symmetry

$$U\partial_u + \sum_{i=1}^{\infty} p^{i+1}\partial_{p^i} \tag{7.12}$$

For the case under consideration Δ_0 is the wave equation (6.1), Δ_1 is the auxiliary system (6.2) and $U\partial_u = Z$ (7.1). In consequence the functions F^n and G^n, see (7.9), where $F^0 = u_y/c^2$ and $G^0 = u_x$, simplify to

$$F^n = \mathrm{pr}\, v_0\big(F^{n-1}\big) \qquad \text{and} \qquad G^n = \mathrm{pr}\, v_0\big(G^{n-1}\big) . \tag{7.13}$$

8. Fourth example

In the previous example the lost symmetry Z (7.1) of system (6.1) covered by the auxiliary system (6.2) has been recovered by an abstract construction, giving rise to the covering $\widetilde{\Delta}$ (equation (7.11)). Here, another covering is given, which also recovers this lost symmetry.

The following system covers system (6.1) ($u^0 = u$ and $v^0 = v$):

$$\begin{aligned} &u_{yy} = c^2 u_{xx} \\ &v_x^n = u_y^n/c^2 \quad v_y^n = u_\alpha^n \quad p_x^n = u^n/c^2 \quad p_y^n = v^n \quad q_x^n = v^n \quad q_y^n = u^n \quad u_x^{n+1} = p^n \\ &\qquad u_y^{n+1} = q^n \quad n = 0, 1, 2, 3, \ldots . \end{aligned} \tag{8.1}$$

A generalized vector field $\sum_{i=0}^{\infty} U^i\partial_{u^i} + V^i\partial_{v^i} + P^i\partial_{p^i} + Q^i\partial_{q^i}$ is a symmetry of (8.1) if and only if ($U = U^0$)

$$\begin{aligned} &D_y^2(U) = c^2 D_x^2(U) \\ &D_x(V^i) = D_y(U^i)/c^2 \quad D_y(V^i) = D_x(U^i) \qquad D_x(P^i) = U^i/c^2 \quad D_y(P^i) = V^i \\ &D_x(Q^i) = V^i \quad D_y(Q^i) = U^i \qquad D_x(U^{i+1}) = P^i \quad D_y(U^{i+1}) = Q^i \quad i = 0, 1, 2, 3, \ldots \end{aligned} \tag{8.2}$$

holds for all solutions of (8.1).

Computations have shown out that the symmetry $Z = U\partial_u$ (equation (7.1)) can be extended to the symmetry $\sum_{i=0}^{\infty} U^i\partial_{u^i} + V^i\partial_{v^i} + P^i\partial_{p^i} + Q^i\partial_{q^i}$, where

$$\begin{aligned} U^n &= xyu^n + cyu^n_x + Cu^n_y - 2n(xq^n + cv^n) \\ V^n &= -xyv^n + cyv^n_x + Cv^n_y + yq^n - 2nu^n/c + (2n+1)(xp^n - u^{n+1}) \\ P^n &= -xyp^n + cyp^n_x + Cp^n_y + yu^{n+1} + 2(n+1)(xv^{n+1} - q^{n+1}) - (2n+1)q^n/c \\ Q^n &= \ \ xyq^n + cyq^n_x + Cq^n_y - (2n+1)(xu^{n+1} + cp^n)\,. \end{aligned} \tag{8.3}$$

Because the auxiliary system (6.2) is covered by the system (8.1) and Z extends to a symmetry of (8.1), the lost symmetry Z turns out to correspond to a non-local symmetry of the auxiliary system.

9. Comment on infinite-dimensional coverings

One might question the infinite-dimensional coverings and restrict oneself to finite-dimensional coverings only. However, infinite-dimensional coverings arise from the same natural geometrical point of view that gave rise to the generalized symmetries.

To show this we consider a generalized vector field, i.e. an expression in the form

$$v = \sum_{i=1}^{p} \xi^i \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q} \varphi_\alpha \frac{\partial}{\partial u^\alpha} \tag{9.1}$$

where ξ^i and φ_α are defined on some jet bundle J^k, k finite. If ξ^i and φ^α are defined on J^0, then this generalized vector field is a vector field on J^0. The corresponding group action is a point transformation on J^0 and therefore symmetries in this form are called point symmetries.

If the generalized vector field is not a point vector field, then it is not a geometric object of J^0. Yet we can prolong the vector field and see if its kth-order prolongation turns out to be a vector field of J^k for some k finite:

$$\mathrm{pr}^{(k)}v = v + \sum_{\alpha=1}^{q}\sum_{|I|=1}^{k} Q^I_\alpha \frac{\partial}{\partial u^\alpha_I} \qquad Q^I_\alpha = \mathrm{D}^I\left(\varphi_\alpha - \sum_{i=1}^{p} \xi^i u^\alpha_i\right) + \sum_{i=1}^{p} \xi^i u^\alpha_{I,i}\,. \tag{9.2}$$

As it happens, some generalized vector fields correspond to vector fields on J^1.

Yet not all generalized vector fields are recovered in this way as geometrical objects. For other k finite, no more generalized vector fields are recovered as geometrical objects of the kth-order jet bundle, as proven by Bäcklund (1876). Yet all generalized vector fields correspond to well defined vector fields on the infinite-order jet bundle J^∞. Thus, in general, a generalized vector field is a geometric object of J^∞.

Now we will consider a non-local symmetry, or more precisely, a non-local solution, i.e.

$$v = \sum_{i=1}^{p} \xi^i \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q} \varphi_\alpha \frac{\partial}{\partial u^\alpha} \tag{9.3}$$

where ξ^i and φ_α depend on a finite number of local and non-local variables. Now we want to recover this expression by a geometric object, i.e. a well defined vector field of some jet bundle. As before, in general all differential consequences of the variables considered, local and non-local, need to be included.

However, the components of the non-local variables have to be constructed as well according to prolongation. As it happens, in general, new non-local variables have to be involved in order to obtain those components. This resembles the case of the generalized vector fields because, *in general, components of derivatives involve higher-order derivatives*, which then have to be included as well.

A typical example of the introduction of an infinite number of non-local variables in order to recover a non-local symmetry has been obtained in section 5. Also an infinite number of conservation laws of the Korteweg–DeVries equation $u_t = uu_1 + u_3$, $u_1 = u_x$ and $u_3 = u_{xxx}$, arises by recovering the non-local solution

$$\left(3t\left(5u^2u_1 + 10uu_3 + 20u_1u_2 + 6u_5\right) + 6x\left(uu_1 + u_3\right) + 2pu_1 + 8\left(u^2 + 3u_2\right)\right)\partial_u . \qquad (9.4)$$

The non-local variable p corresponds to the conservation law $u_t = \left(u^2/2 + u_2\right)_x$, i.e. $p_x = u$ and $p_t = u^2/2 + u_2$.

10. Conclusion

We have shown that the theory of coverings set up by Krasil'shchik and Vinogradov (1989) offers the natural setting for the introduction of non-local symmetries. Potential symmetries, i.e. non-local symmetries introduced by Bluman and Kumei (1989) and Bluman and Reid (1988), give rise to loss of symmetries. Four examples of coverings have been given, in which symmetries lost have been recovered, demonstrating the power of the theory of coverings.

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